

Harnack Inequality for Semilinear SPDE with Multiplicative Noise*

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Abstract

By a new approximate method, dimensional free Harnack inequalities are established for a class of semilinear stochastic differential equations in Hilbert space with multiplicative noise. These inequalities are applied to study the strong Feller property for the semigroup and some properties of invariant measure.

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1 Introduction and main results

The main aim of this paper is to prove Harnack inequality for semilinear stochastic equations on Hilbert spaces with multiplicative noise. This type of inequality, which was proved for the first time in [15], has become a powerful tool in infinite dimensional stochastic analysis. There are many papers prove this type of inequality for SPDE with additive noise, see [3, 5, 7, 9, 10, 11, 12, 16, 17, 18, 19] and reference therein. In [14], the log-Harnack inequality for semilinear SPDE with non-additive noise was proved for the first time, but by the gradient estimate method used there, only determine and time independent coefficient was treated. A new method to deal with the case of general coefficients for SDE was introduced in [17]. This method has been generalized to functional stochastic differential equations, see [20]. In this paper, we generalized this method to the case of semilinear SPDE. There are some disadvantages for finite dimension approximate method here, see Remark 1.3, therefore we use the coupling argument again as in [17] with a slight modification. Since it seems not so clear to solves the similar equation of process Y_t (see equation (2.3) in [17]) in infinite dimension, we turn to a new process which plays the role as the difference of the coupling processes, we get it as a local strong solution of a SPDE and solve the equation

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by truncation in the same sprite in [2]. By this process and Girsanov theorem, we get a coupling in a new probability space. On the other hand, we get Harnack inequality by another type of approximation. We perturb the linear term by a suitable linear operator which closely relates to diffusion term. It's different from finite dimensional approximate and Yosida approximate, by this perturbation, we get a stronger linear term and it makes us to prove the inequality for the perturbed equation more easy.

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, consider the following stochastic differential equation on H :

$$(1.1) \quad dx_t = -Ax_t dt + F(t, x_t)dt + B(t, x_t)dW_t$$

$W = W(t), t \geq 0$ is a cylindrical Brownian motion on H with covariance operator I on filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, and the coefficients satisfy the following hypotheses:

(H1) A is a negative self adjoint operator with discrete spectrum:

$$(1.2) \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty,$$

$\{\lambda_n, n \in \mathbb{N}\}$ are the eigenvalues of A , and $\{e_n\}_{n=1}^{+\infty}$ are the corresponding eigenvectors, the compact C_0 semigroup generated by $-A$ denoted by $S(t)$.

(H2) $F : [0, \infty) \times \Omega \times H \rightarrow H$ and $B : [0, \infty) \times \Omega \times H \rightarrow L(H)$ are $\mathcal{P}_\infty \times \mathcal{B}(H)$ measurable, here \mathcal{P}_∞ is predictable σ -algebra on $[0, \infty) \times \Omega$ and $L(H)$ is all the bounded operators on H , and there exists an increasing function $K_1 : [0, +\infty) \rightarrow [0, \infty)$, such that

$$(1.3) \quad \|F(t, x) - F(t, y)\| + \|B(t, x) - B(t, y)\|_{HS} \leq K_1(t)\|x - y\|,$$

for all $t \geq 0, x \in H$, \mathbb{P} -a.s, here $\|\cdot\|_{HS}$ denote the Hilber-Schmidt norm, and there exists $r > 1$, such that for all $t > 0$,

$$(1.4) \quad \mathbb{E} \left(\int_0^t \|F(s, 0)\| ds \right)^r < \infty,$$

$$(1.5) \quad \sup_{u \in [0, t]} \int_0^u (\mathbb{E} \|S(u-s)B(s, 0)\|_{HS}^{2r})^{\frac{1}{r}} ds < \infty,$$

(H3) There exist a decreasing function $\rho : [0, \infty) \rightarrow (0, \infty)$, and a bounded self adjoint operator B_0 satisfying that there exists $\{b_n > 0 | n \in \mathbb{N}\}$ such that $B_0 e_n = b_n e_n$ and

$$(1.6) \quad B(t, x)B(t, x)^* \geq \rho(t)^2 B_0^2, \quad \forall x \in H, t \geq 0, \quad \mathbb{P}\text{-a.s.},$$

(H4) $\text{Ran}(B(t, x) - B(t, y)) \subset \mathcal{D}(B_0^{-1})$ holds for all $(t, x) \in [0, \infty) \times H$, \mathbb{P} -a.s., and there exists an increasing function $K_2 : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} 2\langle F(t, x) - F(t, y), B_0^{-2}(x - y) \rangle + \|B_0^{-1}(B(t, x) - B(t, y))\|_{HS}^2 \\ \leq K_2(t) \|B_0^{-1}(x - y)\|^2 \end{aligned}$$

holds for all $x, y \in \mathcal{D}(B_0^{-2})$ and all $t \geq 0$, \mathbb{P} -a.s.,

(H5) There exists an increasing function $K_3 : [0, \infty) \rightarrow (0, \infty)$, such that $\|(B(t, x)^* - B(t, y)^*)B_0^{-2}(x - y)\| \leq K_3(t)\|x - y\|_{H_0}$ holds for all $x, y \in H$, $t \geq 0$ and $x - y \in \mathcal{D}(B_0^{-1})$ almost surely.

Remark 1.1. (1) Under (H1), we can replace $\mathcal{D}(B_0^{-2})$ in (H4) by $\bigcup_n H_n$, where $H_n = \text{span}\{e_1, \dots, e_n\}$.

(2) (H3) equals to that $\text{Ran}(B(t, x)) \supset \text{Ran} B_0$ and $\|B(t, x)^{-1}z\| \leq \rho(t)^{-1}\|B_0^{-1}z\|$, for all $z \in \mathcal{D}(B_0^{-1})$, $t \geq 0$, \mathbb{P} -a.s.,

(3) (H5) will be used as a condition in addition to get Harnack inequality, and by (H4), $B_0^{-1}(B(t, x) - B(t, y))$ is an bounded operator, so in (H5) we only require $x - y \in \mathcal{D}(B_0^{-1})$.

For the proof of Remark 1.1, see Appendix. We state our main result of this paper

Theorem 1.2. If (H1)-(H4) hold, then

$$(1.7) \quad P_T \log f(y) \leq \log P_T f(x) + \frac{K_2(T)\|x - y\|_{H_0}}{2(1 - e^{K_2 T})}, \quad \forall f \in \mathcal{B}_b(H), f \geq 1, x, y \in H, T > 0.$$

If, in addition, (H5) holds, then for $p > (1 + \frac{K_3(T)}{\rho(T)})^2$, $\delta_{p,T} = K_3 \vee \frac{\rho(T)}{2}(\sqrt{p} - 1)$, the Harnack inequality

$$(1.8) \quad (P_T f(y))^p \leq (P_T f^p(x)) \exp \left[\frac{K_2(T)\sqrt{p}(\sqrt{p} - 1)\|x - y\|_{H_0}^2}{4\delta_{p,T}[(\sqrt{p} - 1)\rho(T) - \delta_{p,T}](1 - e^{K_2 T})} \right],$$

holds for all $T > 0$, $x, y \in H$ and $f \in \mathcal{B}_b^+(H)$, where $\|x\|_{H_0}^2 = \sum_{n=0}^{+\infty} b_n^{-1} \langle x, e_n \rangle^2$, $H_0 = \mathcal{D}(B_0^{-1})$.

Remark 1.3. One may use the finite dimension approximate method to get the Harnack inequalities, but here we mention that there are difficulties to overcome and it may not be better than the method used here. Let π_n be the projection from H to H_n , then get the following equation on H_n

$$(1.9) \quad dx_t^n = -A_n x_t^n dt + F_n(t, x_t^n) dt + B_n(t, x_t^n) dW_t^n,$$

where,

$$(1.10) \quad A_n = \pi_n A, \quad F_n = \pi_n F|_{H_n}, \quad B_n = \pi_n B|_{H_n}, \quad W^n = \pi_n W,$$

one may find that after projecting to lower dimension, an invertible operator may become degenerate, for example, an operator has the matrix form, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, under the orthonormal basis $\{e_1, e_2\}$. It's easy to find that it's degenerate after projecting to the subspace generated by e_1 . By (H3), one may replace B by its symmetrization $\sqrt{BB^*}$, but constant may become worse in (H2) and (H4), see remark after theorem 1 in [1], and it seems not easy to get similar estimate for $\sqrt{BB^*}$ as in (H4).

2 Proof of Theorem 1.2

Fixed a time $T > 0$, we focus our discussion on the interval $[0, T]$. In order to prove the main theorem, we need some lemmas, and denote $K_i(T)$ by K_i , $i = 1, 2, 3$, for for simplicity's sake. The first lemma prove the existence and uniqueness of mild solution of the equation (1.1), and give some estimates.

Lemma 2.1. *Under the condition (H1) and (H2), equation (1.1) has a pathwise unique mild solution and*

$$(2.1) \quad \sup_{t \in [0, T]} \mathbb{E} \|x_t\|^r \leq C(r, T)(1 + \mathbb{E} \|x_0\|^r).$$

Proof. The existence part goes along the same lines as that of Theorem 7.4 in [4], if we can prove that there exists $p \geq 2$, such that

$$(2.2) \quad \sup_{t \in [0, T]} \mathbb{E} \left\| \int_0^t e^{-(t-s)A} F(s, x_s) ds \right\|^p < \infty,$$

and

$$(2.3) \quad \sup_{t \in [0, T]} \mathbb{E} \left\| \int_0^t e^{-(t-s)A} B(s, x_s) dW_s \right\|^p < \infty$$

for all H -valued predictable processes x defined on $[0, T]$ satisfying

$$(2.4) \quad \sup_{t \in [0, T]} \mathbb{E} \|x_t\|^p < \infty.$$

In fact, for r in (H2),

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{E} \left\| \int_0^t e^{-(t-s)A} B(s, x_s) dW_s \right\|^r \\ & \leq \sup_{t \in [0, T]} \mathbb{E} \left\| \int_0^t e^{-(t-s)A} (B(s, x_s) - B(s, 0)) dW_s \right\|^r + \sup_{t \in [0, T]} \mathbb{E} \left\| \int_0^t e^{-(t-s)A} B(s, 0) dW_s \right\|^r \\ & \leq C(r, T)(1 + \mathbb{E} \|x_t\|^r) + \left(\frac{r}{2}(r-1) \right)^{\frac{r}{2}} \sup_{t \in [0, T]} \left(\int_0^t (\mathbb{E} \|S(t-s)B(s, 0)\|_{HS}^r)^{\frac{2}{r}} ds \right)^r < \infty. \end{aligned}$$

F is treated similarly, we omit it. Estimate (2.1) follows from Gronwall's lemma. For the uniqueness part. If x_t^1, x_t^2 are mild solutions of equation (1.1), then

$$\begin{aligned} (2.5) \quad & \mathbb{E} \sup_{u \in [0, t]} \|x_u^1 - x_u^2\|^r \leq 2^r T \mathbb{E} \sup_{u \in [0, t]} \int_0^u \|S(u-s)(F(s, x_s^1) - F(s, x_s^2))\|^r ds \\ & \quad + 2^r \mathbb{E} \sup_{u \in [0, t]} \left\| \int_0^u S(u-s)(B(s, x_s^1) - B(s, x_s^2)) dW_s \right\|^r \\ & \leq 2^r T \int_0^t \mathbb{E} \|x_u^1 - x_u^2\|^r ds + C(r, T) \mathbb{E} \int_0^t \|x_s^1 - x_s^2\|^r ds \\ & \leq C(r, T) \int_0^t \mathbb{E} \sup_{u \in [0, s]} \|x_u^1 - x_u^2\|^r ds, \end{aligned}$$

by the second inequality, $\mathbb{E} \sup_{u \in [0, t]} \|x_u^1 - x_u^2\|^r < \infty$, then by Gronwall's lemma, $x_t^1 = x_t^2$, $\forall t \in [0, T]$, \mathbb{P} -a.s.

□

Denote $A_\epsilon = A + \epsilon B_0^{-2}$, $\mathcal{D}(A_\epsilon) = \mathcal{D}(A) \cap \mathcal{D}(B_0^{-2}) \subset \mathcal{D}(B_0^{-2})$, it is a self adjoint operator, the eigenvalues of A_ϵ are $\{\lambda_{n,\epsilon} := \lambda_n + \epsilon b_n^{-2} \mid n \in \mathbb{N}\}$ and the eigenvectors remain $\{e_n \mid n \in \mathbb{N}\}$. In fact, one can define a self adjoint operator \tilde{A} by

$$(2.6) \quad \mathcal{D}(\tilde{A}) = \left\{ x \in H \mid \sum_{n=0}^{+\infty} (\lambda_n + \epsilon b_n^{-2})^2 \langle x, e_n \rangle^2 < +\infty \right\},$$

$$(2.7) \quad \tilde{A}x = \sum_{n=0}^{+\infty} (\lambda_n + \epsilon b_n^{-2}) \langle x, e_n \rangle e_n,$$

then by basic inequality and spectral decomposition of A and B_0^{-2} , it is easy to see that $\tilde{A} = A_\epsilon$.

Lemma 2.2. *For the mild solution of equation*

$$(2.8) \quad dx_t^\epsilon = -(A + \epsilon B_0^{-2})x_t^\epsilon dt + F(t, x_t^\epsilon)dt + B(t, x_t^\epsilon)dW_t, \quad x_0^\epsilon = x,$$

we have

$$(2.9) \quad \lim_{\epsilon \rightarrow 0^+} \mathbb{E} \|x_t - x_t^\epsilon\|^2 = 0, \quad \forall t \in [0, T].$$

Proof. Since

$$(2.10) \quad x_t = e^{-tA}x + \int_0^t e^{-(t-s)A}F(s, x_s)ds + \int_0^t e^{-(t-s)A}B(s, x_s)dW_s,$$

$$(2.11) \quad x_t^\epsilon = e^{-t(A+\epsilon B_0^{-2})}x + \int_0^t e^{-(t-s)(A+\epsilon B_0^{-2})}F(s, x_s^\epsilon)ds + \int_0^t e^{-(t-s)(A+\epsilon B_0^{-2})}B(s, x_s^\epsilon)dW_s,$$

then

$$(2.12) \quad \begin{aligned} \|x_t - x_t^\epsilon\|^2 &\leq 3\|(e^{-t\epsilon B_0^{-2}} - 1)e^{(-tA)}x\|^2 \\ &\quad + 3\left\|\int_0^t (e^{-(t-s)A}F(s, x_s) - e^{-(t-s)(A+\epsilon B_0^{-2})}F(s, x_s^\epsilon))ds\right\|^2 \\ &\quad + 3\left\|\int_0^t (e^{-(t-s)A}B(s, x_s) - e^{-(t-s)(A+\epsilon B_0^{-2})}B(s, x_s^\epsilon))dW_s\right\|^2 \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

It's clear that $\lim_{\epsilon \rightarrow 0^+} I_1 = 0$. For I_2 , we have

$$(2.13) \quad \begin{aligned} I_2 &\leq 6T \int_0^t \|(e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})})F(s, x_s)\|^2 ds \\ &\quad + 6T \int_0^t \|e^{-(t-s)(A+\epsilon B_0^{-2})}(F(s, x_s) - F(s, x_s^\epsilon))\|^2 ds =: I_{2,1} + I_{2,2}, \end{aligned}$$

Since

$$(2.14) \quad \|(e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})})F(s, x_s)\| \leq C(1 + \|x_s\|),$$

$$(2.15) \quad \lim_{\epsilon \rightarrow 0^+} \|e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})}\| = 0.$$

By domain convergence theorem $\lim_{\epsilon \rightarrow 0^+} \mathbb{E}I_{2,1} = 0$. On the other hand,

$$\begin{aligned}
(2.16) \quad I_{2,2} &\leq 6T \int_0^t \|e^{-(t-s)(A+\epsilon B_0^2)}(F(s, x_s) - F(s, x_s^\epsilon))\|^2 ds \\
&\leq 6T \int_0^t \|F(s, x_s) - F(s, x_s^\epsilon)\|^2 ds \leq 6TK_1 \int_0^t \|x_s - x_s^\epsilon\|^2 ds.
\end{aligned}$$

For I_3 ,

$$\begin{aligned}
(2.17) \quad \mathbb{E}I_3 &\leq 6\mathbb{E}\left\|\int_0^t (e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})})B(s, x_s) dW_s\right\|^2 \\
&\quad + 6\mathbb{E}\left\|\int_0^t e^{-(t-s)(A+\epsilon B_0^{-2})}(B(s, x_s) - B(s, x_s^\epsilon)) dW_s\right\|^2 = I_{3,1} + I_{3,2},
\end{aligned}$$

and

$$\begin{aligned}
(2.18) \quad \mathbb{E}I_{3,1} &\leq 12T\mathbb{E}\left\|\int_0^t (I - e^{-(t-s)\epsilon B_0^{-2}})(e^{-(t-s)A}B(s, 0)) dW_s\right\|^2 \\
&\quad + 12T\mathbb{E}\left\|\int_0^t (e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})})(B(s, x_s) - B(s, 0)) dW_s\right\|^2 \\
&\leq 12T\mathbb{E}\int_0^t \|(I - e^{-(t-s)\epsilon B_0^{-2}})(e^{-(t-s)A}B(s, 0))\|_{HS}^2 ds \\
&\quad + 12T\mathbb{E}\int_0^t \|(I - e^{-(t-s)\epsilon B_0^{-2}})(e^{-(t-s)A}(B(s, x_s) - B(s, 0)))\|_{HS}^2 ds \\
&=: I_{3,1,1} + I_{3,1,2},
\end{aligned}$$

since

$$(2.19) \quad \|(I - e^{-(t-s)\epsilon B_0^{-2}})e^{-(t-s)A}B(s, 0)\|^2 = \sum_{n=1}^{+\infty} \|(e^{-(t-s)\epsilon B_0^{-2}} - I)e^{-(t-s)A}B(s, 0)e_n\|^2$$

and

$$(2.20) \quad \lim_{\epsilon \rightarrow 0} \|(e^{-(t-s)\epsilon B_0^{-2}} - 1)e^{-(t-s)A}B(s, 0)e_n\| = 0$$

$$(2.21) \quad \|(e^{-(t-s)\epsilon B_0^{-2}} - I)e^{-(t-s)A}B(s, 0)e_n\| \leq \|e^{-(t-s)A}B(s, 0)e_n\|$$

$$(2.22)$$

and by (H2)

$$(2.23) \quad \mathbb{E} \int_0^t \sum_{n=1}^{+\infty} \|e^{-(t-s)A}B(s, 0)e_n\|^2 ds = \mathbb{E} \int_0^t \|e^{-(t-s)A}B(s, 0)\|_{HS}^2 ds < \infty.$$

By dominate convergence theorem, $\lim_{\epsilon \rightarrow 0} I_{3,1,1} = 0$, Note that $B(s, x_s) - B(s, 0) \in L_{HS}(H)$, and

$$\begin{aligned}
(2.24) \quad &\|(I - e^{-(t-s)\epsilon B_0^2})e^{-(t-s)A}(B(s, x_s) - B(s, 0))\|_{HS}^2 \\
&= \sum_{n=1}^{+\infty} \|(I - e^{-(t-s)\epsilon B_0^2})e^{-(t-s)A}(B(s, x_s) - B(s, 0))e_n\|^2
\end{aligned}$$

and

$$(2.25) \quad \|(I - e^{-(t-s)\epsilon B_0^{-2}})(e^{-(t-s)A}(B(s, x_s) - B(s, 0)))e_n\|^2 \leq \|(B(s, x_s) - B(s, 0))e_n\|^2$$

$$(2.26) \quad \mathbb{E} \int_0^t \sum_{n=1}^{+\infty} \|(B(s, x_s) - B(s, 0))e_n\|^2 ds \leq \mathbb{E} \int_0^t \|x_s\|^2 ds < \infty,$$

by dominate convergence theorem, $\lim_{\epsilon \rightarrow 0} \mathbb{E} I_{3,1} = 0$. Finally,

$$(2.27) \quad \mathbb{E} I_{3,2} \leq 6T \mathbb{E} \int_0^t \|B(s, x_s) - B(s, x_s^\epsilon)\|_{HS}^2 ds \leq 6TK_2 \mathbb{E} \int_0^t \|x_s - x_s^\epsilon\|^2 ds.$$

Now, we have

$$(2.28) \quad \mathbb{E} \|x_t - x_t^\epsilon\|^2 \leq \psi_\epsilon(t) + C(T, K_2) \mathbb{E} \int_0^t \|x_s - x_s^\epsilon\|^2 ds$$

for some $\psi_\epsilon(t)$, which satisfies $\lim_{\epsilon \rightarrow 0} \psi_\epsilon(t) = 0$, then by Gronwall's lemma,

$$(2.29) \quad \lim_{\epsilon \rightarrow 0} \mathbb{E} \|x_t - x_t^\epsilon\|^2 = 0, \quad \forall t \in [0, T].$$

□

Firstly, we shall consider the following equation, $\xi_t = \frac{2-\theta}{K_2}(1 - e^{K_2(t-T)})$,

$$(2.30) \quad \begin{aligned} dz_t = & -A_\epsilon z_t dt + (F(t, x_t) - F(t, x_t - z_t))dt + (B(t, x_t) - B(t, x_t - z_t))dW_t \\ & - \frac{1}{\xi_t}(B(t, x_t - z_t) - B(t, x_t))B(t, x_t)^{-1}z_t dt - \frac{1}{\xi_t}z_t dt, \quad z_0 = z. \end{aligned}$$

Note that, by (H2)–(H4),

$$(2.31) \quad F(t, x_t) - F(t, x_t - z_t) \in H, \quad (B(t, x_t) - B(t, x_t - z_t)) \in L_{HS}(H, H_0),$$

$$(2.32) \quad (B(t, x_t - z_t) - B(t, x_t))B(t, x_t)^{-1} \in L(H_0, H_0),$$

it's natural to solve the equation in H_0 , we shall search a suitable Gelfand triple. To this end, we should restrict the operator A_ϵ to H_0 .

Lemma 2.3. *Define $A_{0,\epsilon}$ as follows*

$$(2.33) \quad \mathcal{D}(A_{0,\epsilon}) = B_0(\mathcal{D}(A_\epsilon)), \quad A_{0,\epsilon}x = A_\epsilon x, \quad \forall x \in B_0(\mathcal{D}(A_\epsilon)),$$

then, $A_{0,\epsilon}$ is well defined and $(A_{0,\epsilon}, B_0(\mathcal{D}(A_\epsilon))) = (B_0 A_\epsilon B_0^{-1}, B_0(\mathcal{D}(A_\epsilon)))$.

Proof. It's well defined. In fact for all $x \in B_0(\mathcal{D}(A_\epsilon))$,

$$(2.34) \quad \sum_{n=1}^{+\infty} \lambda_{n,\epsilon} \langle x, e_n \rangle^2 = \sum_{n=0}^{+\infty} \lambda_{n,\epsilon}^2 b_n^2 \langle B_0^{-1}x, e_n \rangle^2 \leq \|B\|_H^2 \sum_{n=1}^{+\infty} (\lambda_{n,\epsilon}^2) \langle B_0^{-1}x, e_n \rangle^2 < +\infty,$$

then $x \in \mathcal{D}(A_\epsilon)$, and

$$(2.35) \quad \sum_{n=1}^{+\infty} b_n^{-2} \langle A_\epsilon x, e_n \rangle^2 = \sum_{n=1}^{+\infty} \lambda_{n,\epsilon}^2 \langle B_0^{-1}x, e_n \rangle^2 < +\infty,$$

then $A_\epsilon x \in \mathcal{D}(B_0^{-1})$, $\forall x \in B_0(\mathcal{D}(A_\epsilon))$, i.e. $A_\epsilon x \in H_0$. Finally, for all $x \in B_0(\mathcal{D}(A))$,

$$(2.36) \quad B_0 A_\epsilon B_0^{-1} x = A_\epsilon B_0 B_0^{-1} x = A_\epsilon x = A_{0,\epsilon} x.$$

□

Now, we can define our Gelfand triple. Let

$$(2.37) \quad (V, \|\cdot\|_V) = (\mathcal{D}(A_{0,\epsilon}^{\frac{1}{2}}), \|A_{0,\epsilon}^{\frac{1}{2}} \cdot\|_{H_0}),$$

then $(V^*, \|\cdot\|_{V^*})$ is the complete of $(H_0, \|A_{0,\epsilon}^{-\frac{1}{2}} \cdot\|_{H_0})$, $V^* \supset H_0 \supset V$ is the triple we need. Since $\mathcal{D}(A_\epsilon) \subset \mathcal{D}(B_0^{-2})$, $\mathcal{D}(A_{0,\epsilon}) \subset \mathcal{D}(B_0^{-3})$, we have the following relationship moreover

$$(2.38) \quad V^* \supset H \supset H_0 \supset \mathcal{D}(B_0^{-2}) \supset V.$$

Lemma 2.4. *If conditions (H1)-(H4) hold, equation (2.30) has a unique strong solution up to the explosion time τ .*

Proof. Let

$$(2.39) \quad G_n(t, v) = \begin{cases} B(t, x_t)^{-1} v, & \|v\|_{H_0} \leq n, \\ B(t, x_t)^{-1} \frac{nv}{\|v\|_{H_0}}, & \|v\|_{H_0} > n, \end{cases}$$

and for simplicity's sake, we denote

$$F(t, x_t - v_1) - F(t, x_t - v_2), \quad G_n(t, v_1) - G_n(t, v_2), \quad B(t, x_t) - B(t, x_t - z_t)$$

by $F(t, v_2, v_1)$, $G_n(t, v_1, v_2)$, $\hat{B}(t, z_t)$ respectively. We consider the following equation firstly,

$$(2.40) \quad \begin{aligned} dz_t &= -A_{0,\epsilon} z_t dt + F(t, z_t, 0) dt - \frac{1}{\xi_t} z_t dt + \frac{1}{\xi_t} \hat{B}(t, z_t) G_n(t, z_t) dt + \hat{B}(t, z_t) dW_t \\ &=: A_{n,\epsilon}(t, z_t) dt + \hat{B}(t, z_t) dW_t \end{aligned}$$

It's clearly that the hemicontinuous holds, since $G_n(t, \cdot)$ remains a Lipschitz mapping from H_0 to H . By the direct calculus, see Appendix, we get that, for all $v, v_1, v_2 \in V$,

(A1) Local monotonicity

$$\begin{aligned} & 2_{V^*} \langle A_{n,\epsilon}(t, v_1) - A_{n,\epsilon}(t, v_2), v_1 - v_2 \rangle_V + \|\hat{B}(t, v_2) - \hat{B}(t, v_1)\|_{L_{HS}(H, H_0)}^2 \\ & \leq \left[K_2 + \frac{2n\sqrt{K_2} - 2}{\xi_t} + \frac{n^2 K_1 \|B_0\|^2}{\epsilon^2 \xi_t^2 \delta^2} + \frac{2}{\xi_t} (\sqrt{K_2} \|v_2\|_{H_0}^2 + \sqrt{\frac{2K_1}{\epsilon}} \|B_0\| \cdot \|v_2\|_V^2) \right] \times \\ & \quad \times \|v_1 - v_2\|_{H_0}^2 - 2(1 - \delta^2) \|v_1 - v_2\|_V^2, \quad \forall \delta \in (0, 1). \end{aligned}$$

(A2) Coercivity

$$\begin{aligned} & 2_{V^*} \langle A_{n,\epsilon}(t, v), v \rangle_V + \|\hat{B}(t, v)\|_{L_{HS}(H, H_0)}^2 \\ & \leq -2(1 - \delta^2) \|v\|_V^2 + \left(\frac{n\sqrt{K_2} - 2}{\xi_t} + \frac{n^2 K_1}{\epsilon^2 \xi_t^2 \delta^2} \right) \|v\|_{H_0}^2, \quad \forall \delta \in (0, 1). \end{aligned}$$

(A3) Growth

$$\|A_{n,\epsilon}(t, v)\|_{V^*}^2 \leq \left(\frac{\|B_0\|^2}{\epsilon \xi_t} K_2 + \left(1 + \frac{\|B_0\|^4 K_1}{\epsilon \xi_t^2} \right) \|v\|_V^2 \right) (1 + \|v\|_{H_0}^4).$$

Since

$$(2.41) \quad \|\hat{B}(t, v)\|_{L_{HS}}^2 = \|B_0^{-1} \hat{B}(t, v)\|_{HS}^2 \leq K_2 \|v\|_{H_0}^2 + \frac{2K_1}{\epsilon} \|B_0\|^3 \|v\|_V \|v\|_{H_0}.$$

does not satisfies the condition (1.2) in [6], but by the basic inequality one can check that the proof in Lemma2.2 goes on well, see Appendix B. By the estimates above and Theorem 1.1 in [6] for any $T_0 < T$, equation (2.40) has unique strong solution $(z_t^n)_{t \in [0, T_0]}$, one can extends the solution to the interval $[0, T)$ by the pathwise uniqueness and continuous. Next we shall let n goes to infinite. Let, $m > n$,

$$(2.42) \quad \tau_m^n = \inf\{t \in [0, T) \mid \|z_t^m\|_{H_0} > n\},$$

define $\inf \emptyset = T$, then

$$(2.43) \quad \begin{aligned} z_t^m = & z_0 + \int_0^t (-A_{0,\epsilon} z_s^m + F(s, z_s^m, 0) - \frac{1}{\xi_s} z_s^m) ds \\ & - \int_0^t \frac{1}{\xi_s} \hat{B}(s, z_s^m) B(s, x_s)^{-1} z_s^m ds + \int_0^t \hat{B}(s, z_s^m) dW_s, \quad t < \tau_m^n, \end{aligned}$$

by Itô's formula and (A1), for $t < \tau_n^n \wedge \tau_m^n$, we have

$$\begin{aligned} & d\|z_t^n - z_t^m\|_{H_0}^2 - 2\langle \hat{B}(t, z_t^n) - \hat{B}(t, z_t^m) dW_t, z_t^n - z_t^m \rangle_{H_0} \\ & = 2_{V^*} \langle A_{n,\epsilon}(t, z_t^n) - A_{n,\epsilon}(t, z_t^m), z_t^n - z_t^m \rangle_V + \|\hat{B}(t, z_t^n) - \hat{B}(t, z_t^m)\|_{L_{HS}(H, H_0)}^2 dt \\ & \leq \left(K_2 + \frac{2}{\xi_t} (n\sqrt{K_1} + \sqrt{K_2} \|z_t^n\|_{H_0}^2 + \sqrt{\frac{2K_1}{\epsilon}} \|B_0\| \cdot \|z_t^n\|_V^2) + \frac{n^2 K_1}{\epsilon^2 \xi_t^2 \delta^2} \|B_0\|^2 \right) \|z_t^n - z_t^m\|_{H_0}^2 \end{aligned}$$

define

$$(2.44) \quad \Psi_s = K_2 + \frac{2}{\xi_s} (\sqrt{K_2} \|z_s^n\|_{H_0}^2 + n\sqrt{K_1} + \sqrt{\frac{2K_1}{\epsilon}} \|B_0\|^2 \|z_s^n\|_V^2) + \frac{n^2 K_1 \|B_0\|^2}{\epsilon^2 \xi_s^2 \delta^2},$$

then

$$(2.45) \quad \begin{aligned} & \exp \left[- \int_0^t \Psi_s ds \right] \|z_t^n - z_t^m\|_{H_0}^2 \\ & \leq \int_0^t 2 \exp \left[- \int_0^r \Psi_s ds \right] \langle (\hat{B}(r, z_r^n) - \hat{B}(r, z_r^m)) dW_r, z_r^n - z_r^m \rangle_{H_0}, \end{aligned}$$

therefore

$$(2.46) \quad \mathbb{E} \left\{ \exp \left[- \int_0^{t \wedge \tau_n^n \wedge \tau_m^n} \Psi_s ds \right] \|z_{t \wedge \tau_n^n \wedge \tau_m^n}^n - z_{t \wedge \tau_n^n \wedge \tau_m^n}^m\|_{H_0}^2 \right\} = 0.$$

Note that

$$(2.47) \quad \mathbb{E} \int_0^t \|z_s^n\|_V^2 ds < \infty, \quad \forall t < T$$

implies

$$(2.48) \quad \int_0^t \|z_s^n\|_V^2 ds < \infty, \quad \forall t \in [0, T), \quad \mathbb{P}\text{-a.s.},$$

then

$$(2.49) \quad z_{t \wedge \tau_n^n \wedge \tau_m^n}^n = z_{t \wedge \tau_n^n \wedge \tau_m^n}^m, \quad \forall t \in [0, T), \quad \mathbb{P}\text{-a.s.},$$

let $t \uparrow T$, by the continuity, we have

$$(2.50) \quad z_{\tau_n^n \wedge \tau_m^n}^n = z_{\tau_n^n \wedge \tau_m^n}^m, \quad \mathbb{P}\text{-a.s.}$$

If $\tau_n^n < \tau_m^n$, $z_{\tau_n^n}^n = z_{\tau_n^n}^m \in \partial B_n^{H_0}(0)$, by the definition of τ_m^n , it's a contradictory. Thus $\tau_n^n \geq \tau_m^n$, similarly, $\tau_n^n \leq \tau_m^n$, so $\tau_n^n = \tau_m^n$, $\mathbb{P}\text{-a.s.}$ and $z_{\tau_n^n}^n = z_{\tau_m^n}^m$. Therefore, we can definite

$$(2.51) \quad z_t = z_t^n, \quad t < \tau_n^n; \quad \tau = \sup_n \tau_n^n,$$

(z, τ) is a strong solution of equation (2.30). By the same method, we can prove the uniqueness easily. \square

Proof of Theorem 1.2. Let

$$\begin{aligned} d\tilde{W}_s &= dW_s + \frac{1}{\xi_s} B(s, x_s)^{-1} z_s ds, \quad s < T \wedge \tau \\ R_s &= \exp \left[- \int_0^s \xi_t^{-1} \langle B(t, x_t)^{-1} z_t, dW_t \rangle - \frac{1}{2} \int_0^s \frac{\|B(t, x_t)^{-1} z_t\|^2}{\xi_t} dt \right], \quad s < T \wedge \tau, \\ \tau_n &= \inf \{t \in [0, T) \mid \|z_t\|_{H_0} > n\}, \quad \mathbb{Q} := R_{T \wedge \tau} \mathbb{P}, \end{aligned}$$

write the equation of z in the form of \tilde{W} :

$$(2.52) \quad dz_t = -A_{0,\epsilon} z_t dt + F(t, z_t, 0) dt + \hat{B}(t, z_t) d\tilde{W}_t - \frac{1}{\xi_t} z_t dt,$$

By Itô's formula and (H4), for $s \in [0, T)$, and for $t < \tau_n \wedge s$,

$$\begin{aligned} d\|z_t\|_{H_0}^2 &= -2\|z_t\|_V^2 dt + 2V^* \langle F(t, z_t, 0), z_t \rangle_V dt - \frac{2\|z_t\|_{H_0}^2}{\xi_t} dt \\ &\quad + \|\hat{B}(t, z_t)\|_{L_{HS}(H, H_0)}^2 dt + 2\langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle_{H_0} \\ &\leq 2\langle F(t, z_t, 0), B_0^{-2} z_t \rangle dt + \|\hat{B}(t, z_t)\|_{L_{HS}(H, H_0)}^2 dt \\ &\quad - \frac{2\|z_t\|_{H_0}^2}{\xi_t} dt + 2\langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle_{H_0} \\ &\leq -\frac{2\|z_t\|_{H_0}^2}{\xi_t} dt + 2\langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle_{H_0} + K_2 \|z_t\|_{H_0}^2 dt, \end{aligned} \quad (2.53)$$

and

$$\begin{aligned}
(2.54) \quad d \frac{\|z_t\|_{H_0}^2}{\xi_t} &\leq -\frac{2\|z_t\|_{H_0}^2}{\xi_t^2} dt + \frac{K_2}{\xi_t} \|z_t\|_{H_0}^2 dt - \frac{\xi_t'}{\xi_t^2} \|z_t\|_{H_0}^2 dt + \frac{2}{\xi_t} \langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle_{H_0} \\
&= \frac{2 - K_2 \xi_t + \xi_t'}{\xi_t^2} \|z_t\|_{H_0}^2 dt + \frac{2}{\xi_t} \langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle_{H_0} \\
&= \frac{\theta}{\xi_t^2} \|z_t\|_{H_0}^2 dt + \frac{2}{\xi_t} \langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle_{H_0},
\end{aligned}$$

by Girsanov theorem, $(\tilde{W})_{t \leq s \wedge \tau_n}$ is a Wiener process under the probability $\mathbb{Q}_{s,n} := R_{s \wedge \tau_n} \mathbb{P}$, and

$$(2.55) \quad \int_0^{s \wedge \tau_n} \frac{\|z_t\|^2}{\xi_t^2} dt \leq \frac{\|z_0\|_{H_0}^2}{\theta \xi_0} + \int_0^{s \wedge \tau_n} \frac{2}{\theta \xi_t} \langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle_{H_0},$$

then

$$(2.56) \quad \mathbb{E}_{\mathbb{Q}_{s,n}} \int_0^{s \wedge \tau_n} \frac{\|z_t\|^2}{\xi_t^2} dt \leq \frac{\|z_0\|_{H_0}^2}{\theta \xi_0},$$

Since, by (H3)

$$\begin{aligned}
(2.57) \quad \log R_u &= -\int_0^u \xi_t^{-1} \langle B(t, x_t)^{-1} z_t, d\tilde{W}_t \rangle + \frac{1}{2} \int_0^u \frac{\|B(t, x_t)^{-1} z_t\|^2}{\xi_t} dt \\
&\leq -\int_0^u \xi_t^{-1} \langle B(t, x_t)^{-1} z_t, d\tilde{W}_t \rangle + \frac{1}{2\rho(T)^2} \int_0^u \frac{\|z_t\|_{H_0}^2}{\xi_t} dt, \quad u \leq s \wedge \tau_n,
\end{aligned}$$

$$(2.58) \quad \mathbb{E} R_{s \wedge \tau_n} \log R_{s \wedge \tau_n} \leq \frac{\|z_0\|_{H_0}^2}{2\theta \xi_0 \rho(T)^2}, \quad \forall s \in [0, T], n \geq 1.$$

As in [17], we can prove that $\{R_{s \wedge \tau} \mid s \in [0, T]\}$ is a martingale. Since

$$(2.59) \quad \mathbb{E}_{\mathbb{Q}} 1_{[\tau_n \leq t]} \frac{\|z_{t \wedge \tau_n}\|_{H_0}^2}{\xi_{t \wedge \tau_n}} \leq \mathbb{E}_{\mathbb{Q}} \frac{\|z_{t \wedge \tau_n}\|_{H_0}^2}{\xi_{t \wedge \tau_n}} \leq \frac{\|z_0\|_{H_0}^2}{\xi_0},$$

and

$$(2.60) \quad \mathbb{E}_{\mathbb{Q}} 1_{[\tau_n \leq t]} \frac{\|z_{t \wedge \tau_n}\|_{H_0}^2}{\xi_{t \wedge \tau_n}} \geq \frac{n \mathbb{Q}(\tau_n \leq t)}{\xi_0}$$

let n goes to infinite, we have $\mathbb{Q}(\tau_n \leq t) = 0, \forall t \in [0, T]$, then $\mathbb{Q}(\tau = T) = 1$. Now, since $\tau = T$, \mathbb{Q} -a.s., equation (2.52) can be solved up to time T . Let

$$(2.61) \quad \zeta = \inf\{t \in [0, T] \mid \|z_t\|_{H_0} = 0\},$$

we shall prove that $\zeta \leq T$, here we assume $\inf \emptyset = +\infty$. Otherwise, there exists a set Ω_0 , such that $\mathbb{P}(\Omega_0) > 0$, and for any $\omega \in \Omega_0$, $\zeta(\omega) > T$, then by the continuity of path, we have

$$(2.62) \quad \inf_{t \in [0, T]} \|z_t(\omega)\|_{H_0} > 0,$$

so

$$(2.63) \quad \int_0^T \frac{\|z_t\|_{H_0}^2}{\xi_t^2} dt = +\infty,$$

but

$$(2.64) \quad \mathbb{E}_{\mathbb{Q}} \int_0^T \frac{\|z_t\|_{H_0}^2}{\xi_t^2} dt \leq \frac{\|z_0\|_{H_0}^2}{2\rho(T)^2\theta\xi_0} < +\infty,$$

hence, $\zeta \leq T$, \mathbb{Q} -a.s., by the uniqueness of solution of equation (2.52), we have

$$(2.65) \quad z_t \equiv 0, \quad t > \zeta, \quad \mathbb{Q}\text{-a.s.}$$

Thus, $z_T = 0$, \mathbb{Q} -a.s.

Next, we shall construct the coupling. Since under the probability space $(\Omega, \mathcal{F}, R_{\tau \wedge T} \mathbb{P})$, $(\tilde{W}_t)_{t \in [0, T]}$ is a Wiener process, let y be the unique mild solution of the following equation

$$(2.66) \quad dy_t = -A_\epsilon y_t dt + F(t, y_t) dt + B(t, y_t) d\tilde{W}_t, \quad y_0 = y,$$

for x_t , it's the unique solution of the following equation

$$(2.67) \quad dx_t = -A_\epsilon x_t dt + F(t, x_t) dt - \frac{z_t}{\xi_t} dt + B(t, x_t) d\tilde{W}_t, \quad x_0 = x.$$

For the process $x_t - y_t$, it's the mild solution of the following equation

$$(2.68) \quad du_t = -A_\epsilon u_t dt + F(t, u_t, 0) dt + \hat{B}(t, u_t) d\tilde{W}_t - \frac{z_t}{\xi_t} dt,$$

note that z_t is a solution of equation

$$(2.69) \quad dz_t = -A_{0, \epsilon} z_t dt + F(t, z_t, 0) dt + \hat{B}(t, z_t) d\tilde{W}_t - \frac{z_t}{\xi_t} dt,$$

Similar to equation (1.41), one can prove that equation (2.68) has a strong solution in H_0 , since $V^* \supset H \supset H_0$ and $A_{0, \epsilon}$ is the restriction of A_ϵ to H_0 , by the relation ship of variational solution and mild solution and the pathwise uniqueness, then $z_t = x_t - y_t$, $\forall t \in [0, T]$, \mathbb{Q} -a.s.

By the method used in [17], we have log-Harnack inequality for equation (2.8) :

$$(2.70) \quad \begin{aligned} P_T^\epsilon \log f(y) &= \mathbb{E}_{\mathbb{Q}} \log f(y_T^\epsilon) = \mathbb{E} R_{T \wedge \tau} \log f(x_T^\epsilon) \leq \mathbb{E} R_{T \wedge \tau} \log R_{T \wedge \tau} + \log \mathbb{E} f(x_T^\epsilon) \\ &\leq \log P_T^\epsilon f(x) + \frac{\|x - y\|_{H_0}}{2\rho(T)^2\theta\xi_0} = \log P_T^\epsilon f(x) + \frac{K_2\|x - y\|_{H_0}}{2\rho(T)^2\theta(2 - \theta)(1 - e^{K_2T})}, \end{aligned}$$

then by lemma 1.2, let $\epsilon \rightarrow 0$, and choose $\theta = 1$, for $f \in \mathcal{B}_b^+(H)$ and $f \geq 1$,

$$(2.71) \quad P_T \log f(y) \leq \log P_T f(x) + \frac{K_2\|x - y\|_{H_0}}{2\rho(T)^2(1 - e^{K_2T})}.$$

If (H5) holds in addition, by inequality (2.55), we have

$$(2.72) \quad \begin{aligned} &\mathbb{E}_{s, n} \exp \left[h \int_0^{s \wedge \tau_n} \frac{\|z_t\|_{H_0}^2}{\xi_t^2} dt \right] \\ &\leq \exp \left[\frac{h\|x - y\|_{H_0}^2}{\theta\xi_0} \right] \mathbb{E}_{s, n} \exp \left[\frac{2h}{\theta} \int_0^{s \wedge \tau_n} \frac{1}{\xi_t} \langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle \right] \\ &\leq \exp \left[\frac{h\|x - y\|_{H_0}^2}{\theta\xi_0} \right] \mathbb{E}_{s, n} \left(\exp \left[\frac{8h^2 K_3^2}{\theta^2} \int_0^{s \wedge \tau_n} \frac{\|z_t\|_{H_0}^2}{\xi_t^2} dt \right] \right)^{\frac{1}{2}}, \end{aligned}$$

for $h = \frac{\theta^2}{8K_3^2}$, and

$$(2.73) \quad \mathbb{E}_{s,n} \exp \left[\frac{\theta^2}{8K_3^2} \int_0^{s \wedge \tau_n} \frac{\|z_t\|_{H_0}^2}{\xi_t^2} dt \right] \leq \exp \left[\frac{\theta K_2 \|x - y\|_{H_0}^2}{4K_3^2(2 - \theta)(1 - e^{-K_2 T})} \right],$$

Similar to [17], we get that

$$(2.74) \quad \sup_{s \in [0, T]} \mathbb{E} R_{s \wedge \tau}^{1+r} \leq \exp \left[\frac{\theta K_2 (2K_3 + \theta \rho(T)) \|x - y\|_{H_0}}{8K_3^2(2 - \theta)(K_3 + \theta \rho(T))(1 - e^{-K_2 T})} \right]$$

and for $p > (1 + K_3)^2$, $\delta_{p,T} = K_3 \vee \frac{\rho(T)}{2}(\sqrt{p} - 1)$, $f \in \mathcal{B}_b^+(H)$, choose $\theta = \frac{2K_3 \rho(T)}{\sqrt{p} - 1}$,

$$(2.75) \quad (P_T^\epsilon f(y))^p \leq (P_T^\epsilon f^p(x)) \exp \left[\frac{K_2(T) \sqrt{p}(\sqrt{p} - 1) \|x - y\|_{H_0}^2}{4\delta_{p,T}[(\sqrt{p} - 1)\rho(T) - \delta_{p,T}](1 - e^{K_2 T})} \right],$$

by lemma 1.2, let $\epsilon \downarrow 0$, we have

$$(2.76) \quad (P_T f(y))^p \leq (P_T f^p(x)) \exp \left[\frac{K_2(T) \sqrt{p}(\sqrt{p} - 1) \|x - y\|_{H_0}^2}{4\delta_{p,T}[(\sqrt{p} - 1)\rho(T) - \delta_{p,T}](1 - e^{K_2 T})} \right],$$

for $x, y \in H, x - y \in \mathcal{D}(B_0^{-1})$.

□

3 Application

In this section, we give some simple applications of Theorem 1.2.

Corollary 3.1. *Assume that F, B are determined and independent of t and (H1) to (H5) hold. If $\lambda_0 > 0$, $\lambda_0 > K_1^2 + 2K_1$ and $B(0) \in L_{HS}(H)$, then*

- (1) P_t has uniqueness invariant measure μ and has full support on H , $\mu(V) = 1$.
- (2) If $\sup_x \|B(x)\| < \infty$, then $\mu(e^{\epsilon_0 \|\cdot\|_H^2}) < \infty$ for some $\epsilon_0 > 0$.
- (3) If there exists $q > 0$ such that $\inf_n b_n^{2q} \lambda_n^{q-1} > 0$, then μ has full support on H_0 .

Proof. Let $(V, \|\cdot\|_V) = (\mathcal{D}(A^{\frac{1}{2}}), \|A^{\frac{1}{2}} \cdot\|)$. Since $\lambda_0 > 0$ and $B(0) \in L_{HS}(H)$, by (H1), equation (1.1) has strong solution and P_t is Feller semigroup. By Ito's formula and $\lambda_0 > K_1^2 - 2K_1$, there exists a constant $c > 0$ such that

$$d\|x_t\|^2 \leq \left(c - 2\left(1 - \frac{K_1^2 + 2K_1}{\lambda_0}\right) \|x_t\|_V^2 + 2\|F(0)\| \cdot \|x_t\| \right) dt + 2\langle B(x_t) dW_t, x_t \rangle$$

and

$$\begin{aligned} d e^{\epsilon \|x_t\|^2} &\leq \epsilon e^{\epsilon \|x_t\|^2} \left(c - 2\left(1 - \frac{K_1^2 + 2K_1}{\lambda_0}\right) \|x_t\|_V^2 + \frac{\epsilon^2}{4} \|B^*(x_t)x_t\|^2 + 2\|F(0)\| \cdot \|x_t\| \right) dt \\ &\quad + 2\epsilon e^{\epsilon \|x_t\|^2} \langle B(x_t) dW_t, x_t \rangle, \end{aligned}$$

for sufficient small ϵ , by Hölder inequality and noting that $\|\cdot\|_V$ is compact function on H , then by standard argument in Theorem 1.2 in [16], one can prove (1) and (2). For (3), $\inf_n b_n^{2q} \lambda_n^{q-1} > 0$ implies that there exists a constant $c(m) > 0$ such that

$$(3.1) \quad \|\cdot\|_{H_0}^2 \leq c(m) \|\cdot\|^2 + \frac{1}{m} \|\cdot\|_V^2, \quad \forall m \geq 1,$$

by Ito's formula, one can get following inequality,

$$(3.2) \quad d\|x_t(x) - x\|^2 \leq -\|x_t(x) - x\|_V^2 dt + (c_1 + c_2 \|x_t(x)\|^2) dt + 2\langle B(x_t) dW_t, x_t - x \rangle$$

here we denote $x_t(x)$ for the process starts from x , c_1, c_2 are constants depend on x . Using Harnack inequality (1.8), (3) can be proved following the line of [19]. \square

Corollary 3.2. *Assume (H1) to (H5) hold, F and B are determined and time independent, then for any $t > 0$, P_t is H_0 -strong Feller. Let μ be the P_t -subinvariant probability with full support on H_0 as in [14], then the transition density $p_t(x, y)$ w.r.t. μ satisfies*

$$(3.3) \quad \|p_t(x, \cdot)\|_{L^p(\mu)} \leq \left\{ \int_{H_0} \exp \left[-\frac{K_2 \sqrt{q} (\sqrt{q} - 1) \|x - y\|_{H_0}^2}{4\delta_q [(\sqrt{q} - 1)\rho - \delta_q] (1 - e^{K_2 t})} \right] \mu(dy) \right\}^{-\frac{1}{q}}$$

for all $1 < p < \frac{(K_3 + \rho)^2}{(K_3 + \rho)^2 - 1}$, here $q = \frac{p}{p-1}$.

Proof. It follows the proof of [16, 14, 19].

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Appendix

A. Proof of Remark 1.1

Proof of (1): since $\bigcup_n H_n$ is a core of B_0^{-2} , for any $x \in \mathcal{D}(B_0^{-2})$, choose $\{x_n\}$ such that $x_n \rightarrow x$ and $B_0^{-2} x_n \rightarrow B_0^{-2} x$, hence $B_0^{-1} x_n \rightarrow x$, as $n \rightarrow +\infty$. Similarly, a sequence $\{y_n\}$ with the same property. Therefore

$$\begin{aligned} & \|B_0^{-1}[(B(t, x_n) - B(t, y_n)) - (B(t, x_m) - B(t, y_m))]\|_{HS}^2 \\ & \leq 2K_2(\|B_0^{-1}(x_n - x_m)\|^2 + \|B_0^{-1}(y_n - y_m)\|^2) - 4\langle F(t, x_n) - F(t, x_m), B_0^{-2}(x_n - x_m) \rangle \\ & \quad - 4\langle F(t, y_n) - F(t, y_m), B_0^{-2}(y_n - y_m) \rangle, \end{aligned}$$

by the continuous of F , we have that $\{B(t, x_n) - B(t, y_n)\}$ forms a Cauchy sequence in $L_{HS}(H, H_0)$. Note that $B(t, x_n) - B(t, y_n)$ convergent to $B(t, x) - B(t, y)$ in $L_{HS}(H)$, and B_0^{-1} is closed, we have $B(t, x) - B(t, y) \in L_{HS}(H, H_0)$,

$$\lim_{n \rightarrow +\infty} (B(t, x_n) - B(t, y_n)) = B(t, x) - B(t, y),$$

and

$$2\langle F(t, x) - F(t, y), B_0^{-2}(x - y) \rangle + \|B_0^{-1}(B(t, x) - B(t, y))\|_{HS}^2 \leq K_2 \|B_0^{-1}(x - y)\|^2.$$

Proof of (2): we assume $\rho(t) = 1$, by definition, it's clear that B_0 is one to one and has dense range.

$$B(t, x)B(t, x)^* \geq B_0^2 \Leftrightarrow \|B(t, x)^*y\| \geq \|B_0y\|, \forall y \in H,$$

implies that $\text{Ran}B(t, x) \supset \text{Ran}B_0$ by Proposition B.1 in [4], and

$$\|z\| \geq \|B_0(B(t, x)^*)^{-1}z\|, \forall z \in \text{Ran}(B(t, x)^*).$$

Since for any $z \in \text{Ran}(B(t, x)^*)$, $y \in \text{Ran}(B(t, x))$, we have

$$(3.4) \quad \langle B(t, x)^{-1}y, z \rangle = \langle B(t, x)B(t, x)^{-1}y, (B(t, x)^*)^{-1}z \rangle = \langle y, (B(t, x)^*)^{-1}z \rangle,$$

then

$$(3.5) \quad z \in \mathcal{D}((B(t, x)^{-1})^*), (B(t, x)^{-1})^*z = (B(t, x)^*)^{-1}z.$$

On the other hand, for any $z \in \mathcal{D}((B(t, x)^{-1})^*)$, there exists z^* such that

$$(3.6) \quad \langle B(t, x)^{-1}y, z \rangle = \langle y, z^* \rangle, \forall y \in \mathcal{D}((B(t, x)^{-1})),$$

let $u = B(t, x)^{-1}y$, then $\langle u, z \rangle = \langle B(t, x)u, z^* \rangle$, we have $z = B(t, x)^*z^*$ and

$$(B(t, x)^*)^{-1}z = z^* = (B(t, x)^{-1})^*z,$$

hence $\mathcal{D}((B(t, x)^{-1})^*) = \mathcal{D}((B(t, x)^*)^{-1})$. Therefore, $\|z\| \geq \|B_0(B(t, x)^{-1})^*z\|$, for all $z \in \text{Ran}B(t, x)^*$. Since $\text{Ran}(B(t, x)^*)$ is dense in H , $B_0(B(t, x)^{-1})^*$ can be extended to be a bounded operator on H , and for all $z \in H$, $y \in H$, there is $\{z_n\}_{n=1}^{+\infty}$, $\lim_n z_n = z$, such that $\lim_n B_0(B(t, x)^{-1})^*z_n = B_0(B(t, x)^{-1})^*z$, then

$$(3.7) \quad \begin{aligned} \langle B_0(B(t, x)^{-1})^*z, y \rangle &= \lim_n \langle B_0(B(t, x)^{-1})^*z_n, y \rangle \\ &= \lim_n \langle z_n, (B(t, x)^{-1})B_0y \rangle = \langle z, (B(t, x)^{-1})B_0y \rangle, \end{aligned}$$

hence $\|(B(t, x)^{-1})B_0y\| \leq \|y\|$, for all $y \in H$, let $z = B_0y$, then $\|(B(t, x)^{-1})z\| \leq \|B_0^{-1}z\|$, for all $z \in \mathcal{D}(B_0^{-1})$. By Proposition B.1 in [4], and the proof above, the converse is easy. \square

B. For Lemma 2.4

(1) *For local monotonicity.* For any $v_1, v_2 \in V$,

$$(3.8) \quad -2_{V^*} \langle A_{0,\epsilon}(v_1 - v_2), v_2 \rangle_V = -2\|\sqrt{A_{0,\epsilon}}(v_1 - v_2)\|_{H_0}^2 = -2\|v_1 - v_2\|_V^2,$$

$$(3.9) \quad \begin{aligned} &2_{V^*} \langle F(t, v_1, v_2), v_1 - v_2 \rangle_V + \|\hat{B}(t, v_1) - \hat{B}(t, v_2)\|_{L_{HS}(H, H_0)}^2 \\ &= 2\langle F(t, v_1, v_2), B_0^{-2}(v_1 - v_2) \rangle + \|B_0^{-1}(\hat{B}(t, v_1) - \hat{B}(t, v_2))\|_{HS}^2 \\ &\leq K_2\|v_1 - v_2\|_{H_0}^2 \end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\xi_t} {}_{V^*} \langle \hat{B}(t, v_1) G_n(t, v_1) - \hat{B}(t, v_2) G_n(t, v_2), v_1 - v_2 \rangle_V \\
&= \frac{1}{\xi_t} \langle (\hat{B}(t, v_1) - \hat{B}(t, v_2)) G_n(t, v_1) - \hat{B}(t, v_2) G_n(t, v_1, v_2), B_0^{-2}(v_1 - v_2) \rangle \\
(3.10) \quad & \leq \frac{1}{\xi_t} \|B_0^{-1}(\hat{B}(t, v_1) - \hat{B}(t, v_2))\| \cdot \|G_n(t, v_1)\| \cdot \|v_1 - v_2\|_{H_0} \\
& \quad + \frac{1}{\xi_t} \|B_0^{-1} \hat{B}(t, v_2)\| \cdot \|G_n(t, v_1, v_2)\| \cdot \|v_1 - v_2\|_{H_0},
\end{aligned}$$

note that, by (H1),

$$\begin{aligned}
& \|B_0^{-1}(\hat{B}(t, v_1) - \hat{B}(t, v_2))\|_{HS}^2 \leq K_2 \|v_1 - v_2\|_{H_0}^2 - 2 \langle F(t, v_1, v_2), B_0^{-2}(v_1 - v_2) \rangle \\
& \leq K_2 \|v_1 - v_2\|_{H_0}^2 + 2K_1 \|B_0 A_{0,\epsilon}^{-\frac{1}{2}} A_{0,\epsilon}^{\frac{1}{2}}(v_1 - v_2)\|_{H_0} \cdot \|B_0^{-1} A_{0,\epsilon}^{-\frac{1}{2}} A_{0,\epsilon}^{\frac{1}{2}}(v_1 - v_2)\|_{H_0} \\
& \leq K_2 \|v_1 - v_2\|_{H_0}^2 + 2K_1 \left(\sup_n \frac{b_n}{\sqrt{\lambda_n + \epsilon b_n^{-2}}} \right) \left(\sup_n \frac{1}{b_n \sqrt{\lambda_n + \epsilon b_n^{-2}}} \right) \|v_1 - v_2\|_V^2 \\
& \leq K_2 \|v_1 - v_2\|_{H_0}^2 + \frac{2}{\epsilon} K_1 \|B_0\|^2 \|v_1 - v_2\|_V^2,
\end{aligned}$$

hence

$$\begin{aligned}
& \frac{1}{\xi_t} \|B_0^{-1}(\hat{B}(t, v_1) - \hat{B}(t, v_2))\| \cdot \|G_n(t, v_1)\| \cdot \|v_1 - v_2\|_{H_0} \\
(3.11) \quad & \leq \frac{n}{\xi_t} (\sqrt{K_2} \|v_1 - v_2\|_{H_0} + \sqrt{\frac{2}{\epsilon}} K_1 \|B_0\| \cdot \|v_1 - v_2\|_V) \|v_1 - v_2\|_{H_0} \\
& \leq \left(\frac{n}{\xi_t} \sqrt{K_2} + \frac{n^2 K_1 \|B_0\|^2}{\epsilon \xi_t^2 \delta^2} \right) \|v_1 - v_2\|_{H_0}^2 + \delta^2 \|v_1 - v_2\|_V^2,
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\xi_t} \|B_0^{-1} \hat{B}(t, v_2)\| \cdot \|G_n(t, v_1, v_2)\| \cdot \|v_1 - v_2\|_{H_0} \\
(3.12) \quad & \leq \frac{1}{\xi_t} (\sqrt{K_2} \|v_2\|_{H_0} + \sqrt{\frac{2K_1}{\epsilon}} \|B_0\| \cdot \|v_2\|_V) \|v_1 - v_2\|_{H_0}^2,
\end{aligned}$$

therefore, we have

$$\begin{aligned}
& 2 {}_{V^*} \langle A_{n,\epsilon}(t, v_1) - A_{n,\epsilon}(t, v_2), v_1 - v_2 \rangle_V + \|\hat{B}(t, x_t - v_2) - \hat{B}(t, x_t - v_1)\|_{L_{HS}(H, H_0)}^2 \\
& \leq \left[K_2 + \frac{2n\sqrt{K_2} - 2}{\xi_t} + \frac{n^2 K_1 \|B_0\|^2}{\epsilon^2 \xi_t^2 \delta^2} + \frac{2}{\xi_t} (\sqrt{K_2} \|v_2\|_{H_0} + \sqrt{\frac{2K_1}{\epsilon}} \|B_0\| \cdot \|v_2\|_V) \right] \times \\
& \quad \times \|v_1 - v_2\|_{H_0}^2 - 2(1 - \delta^2) \|v_1 - v_2\|_V^2.
\end{aligned}$$

(2) *For coercivity:*

$$(3.13) \quad -2 {}_{V^*} \langle A_{0,\epsilon} v, v \rangle_V = -2 \|v\|_V^2, \quad \|B_0^{-1} \hat{B}(t, v)\|_{HS}^2 + 2 \langle F(t, v, 0), B_0^{-2} v \rangle \leq K_2 \|v\|^2,$$

$$\begin{aligned}
(3.14) \quad \frac{2}{\xi_t} {}_{V^*} \langle \hat{B}(t, v) G_n(t, v), v \rangle_V &\leq \frac{2}{\xi_t} \|B_0^{-1} \hat{B}(t, v)\| \cdot \|G_n(t, v)\| \cdot \|v\|_{H_0} \\
&\leq \frac{2n}{\xi_t} (K_2 \|v\|^2 + \frac{2K_1}{\epsilon} \|B_0\|^2 \|v\|_V^2)^{\frac{1}{2}} \|v\|_{H_0} \\
&\leq \frac{2n}{\xi_t} (\sqrt{K_2} \|v\|_{H_0} + \sqrt{\frac{2K_1}{\epsilon}} \|B_0\| \cdot \|v\|_V) \|v\|_{H_0} \\
&\leq (\frac{2n\sqrt{K_2}}{\xi_t} + \frac{2n^2 K_1 \|B_0\|^2}{\epsilon \xi_t^2 \delta^2}) \|v\|_{H_0}^2 + \delta^2 \|v\|_V^2,
\end{aligned}$$

hence

$$\begin{aligned}
(3.15) \quad &2_{V^*} \langle A_n(t, v), v \rangle_V + \|\hat{B}(t, v)\|_{L_{HS}(H, H_0)}^2 \\
&\leq -2(1 - \delta^2) \|v\|_V^2 + (\frac{n\sqrt{K_2} - 2}{\xi_t} + \frac{n^2 K_1}{\epsilon^2 \xi_t^2 \delta^2}) \|v\|_{H_0}^2.
\end{aligned}$$

(3) *For Growth:*

$$(3.16) \quad \|A_{0,\epsilon} v\|_{V^*}^2 = \|v\|_V^2, \quad \frac{1}{\xi_t} v|_{V^*} = \frac{1}{\xi_t} \|v\|_{V^*}, \quad \|F(t, v, 0)\|_{V^*} \leq \frac{K_1}{\sqrt{\epsilon}} \|v\|,$$

since, by (H1),

$$(3.17) \quad |{}_{V^*} \langle F(t, v, 0), z \rangle_V| = |\langle F(t, v, 0), B_0^{-2} z \rangle| \leq K_1 \|v\| \cdot \|B_0^{-2} z\| \leq \frac{K_1}{\sqrt{\epsilon}} \|v\| \cdot \|z\|_V.$$

And

$$\begin{aligned}
(3.18) \quad \|\frac{1}{\xi_t} \hat{B}(t, v) G_n(t, v)\|_{V^*} &\leq \frac{\|B_0\|}{\sqrt{\epsilon} \xi_t} \|\hat{B}(t, v) G_n(t, v)\|_{H_0} \\
&\leq \frac{\|B_0\|}{\sqrt{\epsilon} \xi_t} \|B_0^{-1} \hat{B}(t, v)\| \cdot \|G_n(t, v)\|_{L(H_0, H)} \\
&\leq \frac{\|B_0\|}{\sqrt{\epsilon} \xi_t} (\sqrt{K_2} \|v\|_{H_0} + \sqrt{\frac{2K_1}{\epsilon}} \|v\|_V \|B_0\|_{H_0}) \|v\|_{H_0},
\end{aligned}$$

we have

$$(3.19) \quad \|A_{n,\epsilon}(t, v)\|_{V^*}^2 \leq \left(\frac{\|B_0\|^2}{\epsilon \xi_t} K_2 + \left(1 + \frac{\|B_0\|^4 K_1}{\epsilon \xi_t^2} \right) \|v\|_V^2 \right) (1 + \|v\|_{H_0}^4).$$

(4) *For the Lemma 2.2 of [6]:* We give new estimates to replace inequalities (2.3) and (2.4) there. For convenience, we use the notations there. In (2.3), we only have to replace $f_s \cdot \|X_s^{(n)}\|_H^{p-2}$ by $\|X_s^{(n)}\|_V \|X_s^{(n)}\|_H \cdot \|X_s^{(n)}\|_H^{p-2}$ and use the basic inequality

$$(3.20) \quad a \cdot b \leq \frac{a^2}{2\delta} + \frac{\delta}{2} b^2, \quad \forall \delta > 0,$$

and note that in our case $\alpha = 2$. For (2.4), one can use the following estimate,

$$\begin{aligned}
& \mathbb{E} \left(\int_0^{\tau_R^n} \|X_s^{(n)}\|_H^{2p-2} \|B(s, X_s^{(n)})\|_2^2 ds \right)^{\frac{1}{2}} \\
& \leq \mathbb{E} \left(\int_0^{\tau_R^n} C \|X_s^{(n)}\|_H^{2p-2} (\|X_s^{(n)}\|_V \|X_s^{(n)}\|_H + \|X_s^{(n)}\|_H^2) ds \right)^{\frac{1}{2}} \\
& \leq C(\delta_1) \mathbb{E} \left(\int_0^{\tau_R^n} \|X_s^{(n)}\|_H^{2p-2} \|X_s^{(n)}\|_H^2 ds \right)^{\frac{1}{2}} + \sqrt{\delta_1} \mathbb{E} \left(\int_0^{\tau_R^n} \|X_s^{(n)}\|_H^{2p-2} \|X_s^{(n)}\|_V^2 ds \right)^{\frac{1}{2}} \\
& \leq \delta_2 \mathbb{E} \sup_{s \in [0, \tau_R^n]} \|X_s^{(n)}\|_H^p + C(\delta_1, \delta_2) \mathbb{E} \int_0^{\tau_R^n} \|X_s^{(n)}\|_H^p ds \\
& \quad + \sqrt{\delta_1} \mathbb{E} \sup_{s \in [0, \tau_R^n]} \|X_s^{(n)}\|_H^{\frac{p}{2}} \left(\int_0^{\tau_R^n} \|X_s^{(n)}\|_H^{p-2} \|X_s^{(n)}\|_V^2 ds \right)^{\frac{1}{2}} \\
& \leq (\delta_2 + \delta_3) \mathbb{E} \sup_{s \in [0, \tau_R^n]} \|X_s^{(n)}\|_H^p + \frac{\delta_1}{4\delta_3} \mathbb{E} \int_0^{\tau_R^n} \|X_s^{(n)}\|_H^{p-2} \|X_s^{(n)}\|_V^2 ds + C(\delta_1, \delta_2) \mathbb{E} \int_0^{\tau_R^n} \|X_s^{(n)}\|_H^p ds,
\end{aligned}$$

choose δ_2, δ_3 small enough and δ_1 such that $\frac{\delta_1}{4\delta_3}$ small enough, using $\alpha = 2$ again, then Gronwall's lemma can be applied as in [6].

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